(Lecture 05) Mortgage Repayment Schedules

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1 The Math of Mortgage Payments

Question: *Let's say that Jon is borrowing* \$500, 000 *with a fixed-rate 30-year mortgage for a new home at* 7%. *If he wants to pays the same amount at a regular schedule each month, how much will Jon have to pay?*

There are two primary factors to consider when answering this question:

- Each month Jon will pay P;
- Each month, the amount remaining will accrue interest at 7%.

This leads to the following recurrence relation. Let t be the number of months since Jon took out the loan, $a(t)$ denote the amount of the loan remaining, r denote the interest rate, and P the fixed monthly payment. Then we have

$$a(t+1) = a(t)\left(1 + \frac{r}{12}\right) - P. \tag{1}$$

This may be solved as follows.

Theorem 1.1. *The closed-form solution to Equation (1) is*

$$a(t) = \frac{P - (P - a(0)(u-1))\,u^t}{u-1} \tag{2}$$

where $u = 1 + \frac{r}{12}$.

Proof. We prove by induction.

Base case. From Equation (1) we have $a(1) = a(0)u - P$. From Equation (2) we have

$$\begin{aligned}
a(1) &= \frac{P - (P - a(0)(u-1))\,u}{u-1} \\
&= \frac{P - (P - a(0)(u-1))\,(u-1) - (P - a(0)(u-1))}{u-1} \\
&= -P + a(0)(u-1) + \frac{P - P + a(0)(u-1)}{u-1} \\
&= a(0)u - P.
\end{aligned}$$

Induction hypothesis. Assume that for any $k \geq 0$, Equation (2) holds for k.

Induction step. By Equation (1) we have

$$\begin{aligned}
a(k+1) &= a(k)u - P \\
&= \left(\frac{P - (P - a(0)(u-1))\,u^k}{u-1}\right)u - P \\
&= \frac{P - (P - a(0)(u-1))\,u^{k+1}}{u-1}.
\end{aligned}$$

This concludes our proof. □

Now let's compute what P should be, given that the loan should be completed in 30 years, or, equivalently, 360 months. If we plug this into Equation (2), we get

$$0 = \frac{P - (P - a(0)(u-1))\,u^{360}}{u-1}$$

which yields

$$P = \frac{a(0)u^{360}(1-u)}{1-u^{360}}.$$

Let's take $r = 0.07$, so that $u = 1 + \frac{0.07}{12} \approx 1.00583$, and $a(0) = 500,000$. This yields

$$P = \frac{500,000 \cdot 1.00583^{360}(-0.00583)}{1 - 1.00583^{360}} \approx 3,326.51.$$

2 Comparison against Online Calculator

Now let's compare our result against https://www.mortgagecalculator.org/. [1]

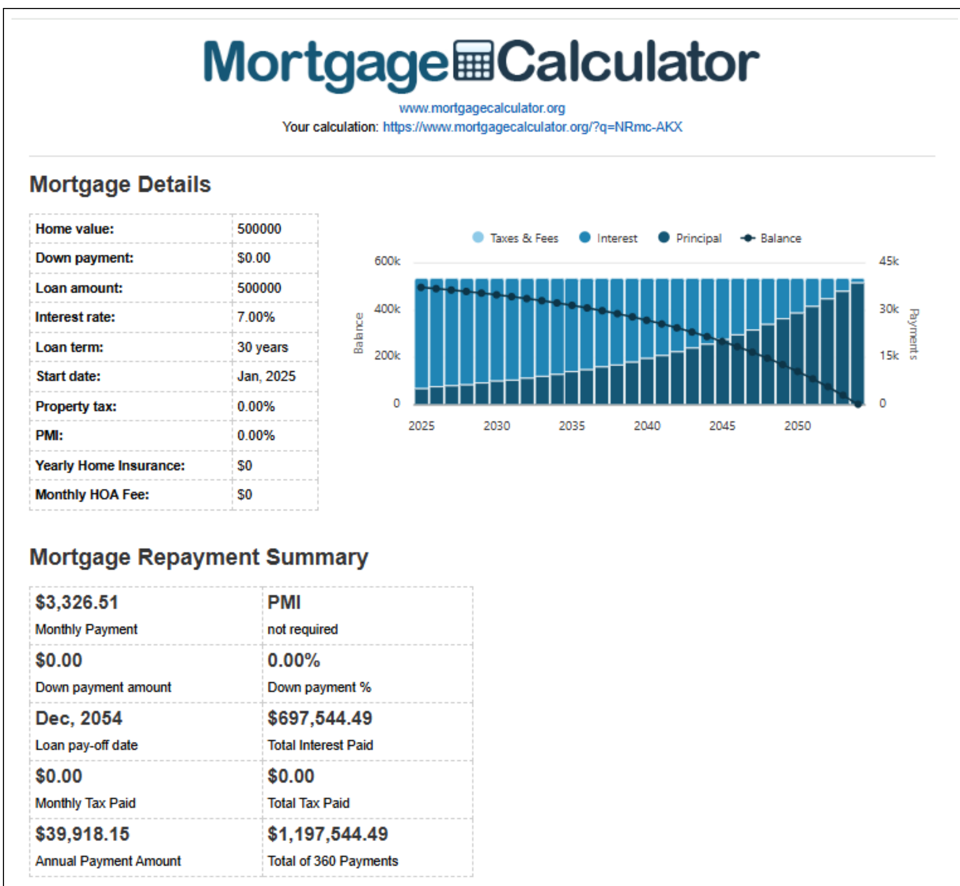

Fig. 1. Mortgage Payment Example. *Source: MortgageCalculator.*

We can see from Figure 1 that our payment is, indeed, $3,326.51 per month. This confirms that our math aligns with that of the online Mortgage Calculator.

There are two other observations we can draw from Figure 1. First, the total payments made over the 30 years is $1,197,544.49. This is extensive and significantly more than the initial

[1] See https://www.mortgagecalculator.org/?q=NRmc-AKX.

$\$500,000$ loan. However, it is significantly less than the return of 30-year bonds, which would yield $\$500,000\,(1+0.07)^{30} = \$3,816,127.52$.

So why would people and institutions invest in mortgages instead of 30-year bonds? This is because, in general, there are no 30-year bonds available for purchase at as high a rate as 30-year mortgages. Figure 2 shows the chart of 30-year fixed rate mortgage averages in the United States (in green) and 30-year treasury yields (in blue) dating back to 1970, according to data from the Federal Reserve Bank of St. Louis. We can see that mortgage rates generally follow the 30-year yield [2] and are significantly above the 30-year yield. The higher yield offered to mortgage investors is due to the greater risk that homeowners default on their loans, as opposed to the probability of the US government defaulting on its treasuries.

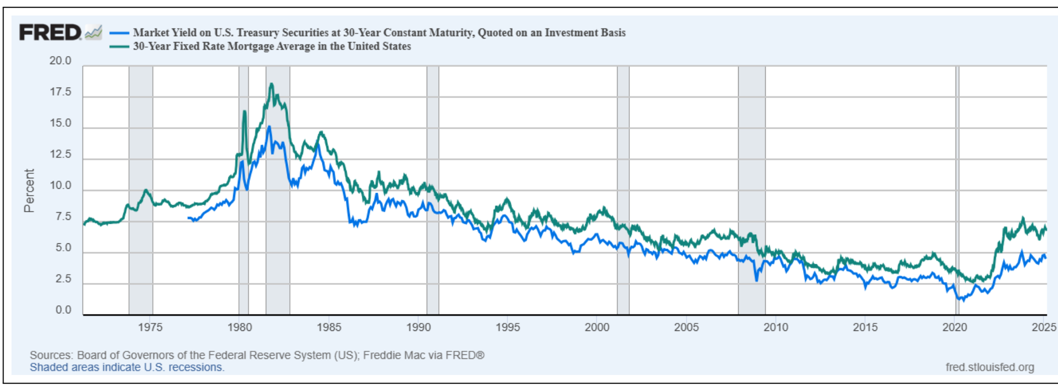

Fig. 2. 30-year Fixed Rate Mortgage Average in the United States. *Source: FRED.*

Our second observation from Figure 1 is that most of Jon's monthly payment goes toward paying interest at the beginning of the loan term, while it only significantly builds Jon's equity in the home toward the end of the loan term. This is a result of the *amortization schedule* of Jon paying the same amount each month (see Figure 3). [3] For example, toward the beginning of the loan, the amount of interest accruing on the remaining balance of the loan is almost as much as the amount that Jon pays down. In the first month, Jon pays $\$3,326.51$ while the remaining balance accrues $(500000 - 3326.51) * \left(\frac{0.07}{12}\right) = 2897.26$. That is, he only pays down $\$429.25$ off the $\$500,000$ loan that first month. It's progress, but very minimal. This concept extends to the *convex* amount of equity Jon builds in the property over time (represented by the dark blue upward curve in Figure 1) and *concave* amount of loan balance remaining over time (represented by the black line downward curve in Figure 1).

Final note: In this lecture, so far, we have assumed \$0/yr in property tax, 0% PMI, \$0/yr in home owners' insurance, and 0% in HOA fees. Property tax, insurance, and HOA fees don't directly affect Jon's loan. They are just additional payments he must make. According to Chase "*Private Mortgage Insurance (PMI) is a type of insurance that lenders require for conventional mortgages with a high loan-to-value (LTV) ratio. Lenders accept some level of risk with these mortgages, and PMI helps to lower that risk.*" [4] This also does not affect Jon's loan payment, in terms of the calculations above, but must be paid until Jon's equity is at least 20% of the initial loan amount.

[2] Some people notice that mortgage rates track closer to the 10-year treasury yield.
[3] See https://www.investopedia.com/terms/a/amortization.asp.
[4] See https://www.chase.com/personal/mortgage/education/financing-a-home/what-is-pmi-calculated.

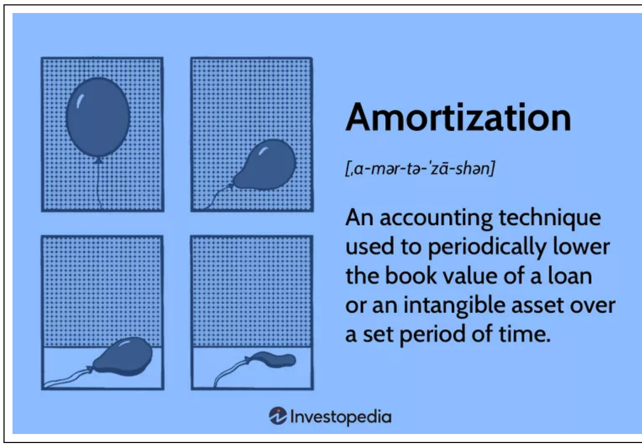

Fig. 3. Definition of Amortization. *Source: Investopedia.*